Section 2.7: Theorems of Polynomial Equations
Essential Question: How do you find and write the equation of a quadratic or cubic function, given the roots?

Fundamental Theorem of Algebra * the # of roots = degree

If p(x) has degree n (n>0) with complex coefficients then p(x) = 0 has exactly n roots, where double roots count as 2, triple roots count as 3 and so on...

Complex Conjugates Theorem

If a + bi is a solution for p(x), so is a−bi

If a + √b is a solution for p(x), so is a−√b

Odd Degree

If p(x) has an odd degree then p(x) has at least one real root

Examples

Factor and find the roots of the following polynomial equations.

1) $x^3 + 6x^2 - 4x - 24 = 0$ ← Degree

$x^2(x+6) - 4(x+6)$

$(x+6)(x^2-4)=0$

$(x+6)(x-2)(x+2)=0$

$\boxed{x=-6 \quad x=2 \quad x=-2}$

*3 roots = degree of 3

2) $x^4 - 7x^2 + 12 = 0$ ← degree

$ax^4+bx^2+c \rightarrow$ factor like a quadratic

a·c = 12, −4 · −3, −7 = b

$(x^2-4)(x^2-3)$

$x^2-4=0 \qquad x^2-3=0$

$x^2=4 \qquad x^2=3$

$x=\pm\sqrt{4} \qquad \boxed{x=\pm\sqrt{3}}$

$\boxed{x=\pm 2}$

2 roots — 2 roots — 4 total roots = degree of 4

SUM and PRODUCT of ROOTS

Given a polynomial...

Sum of the roots $= \boxed{\dfrac{-b}{a}} = \dfrac{\text{coefficient of (n-1) term}}{\text{leading coefficient}}$ (L.C.)

Product of the roots $= \boxed{\dfrac{\text{constant}}{a}}$ ↳ L.C. if the degree is even

$= \boxed{\dfrac{-\text{constant}}{a}}$ ↳ L.C. if the degree is odd

Look @ Ex1

sum $= -6+2+-2 = -6$

$= -\frac{b}{a} = -\frac{6}{1} = -6$ > same

prod $= (-6)(2)(-2) = 24$ — same

$= \frac{-\text{const}}{a} = \frac{+24}{1} = 24$

Examples

Find the sum and the product of the polynomial equations without factoring.

3) $1x^2 + 5x + 6 = 0$ (a, b, c)

Sum $= \boxed{\frac{-b}{a}} = \frac{-5}{1} = \boxed{-5}$

(even) product $= \boxed{\frac{\text{const.}}{a}} = \frac{6}{1} = \boxed{6}$

4) $1x^4 - 11x^2 + 24 = 0$ (a, c, const.)

$0x^3 \to b = 0$ sum $= \boxed{\frac{-b}{a}} = \frac{0}{1} = \boxed{0}$

(even) product $= \boxed{\frac{\text{const.}}{a}} = \frac{24}{1} = \boxed{24}$

5) $2x^3 + 5x^2 - 3x = 0$ (a, b, c, const = 0)

Degree = 3 → odd

Sum $= \boxed{\frac{-b}{a}} = \boxed{\frac{-5}{2}}$

product $= \boxed{\frac{-\text{const}}{a}} = \frac{0}{2} = \boxed{0}$

Example 6

Find a quadratic equation whose roots are 3 ± 5i

Sum $= 3+5i + 3-5i = 6 = \frac{-b}{a} = \frac{6}{1}$ $a = 1$, $b = -6$

even → product $= \frac{\text{constant}}{a} = (3+5i)(3-5i)$

$= 9 - 25i^2$

$= 9 + 25 = \frac{34}{1} = \frac{\text{const}}{a} \to c = 34$

$\boxed{f(x) = 1x^2 - 6x + 34}$

Check: $\frac{6 \pm \sqrt{36 - 4(1)(34)}}{2(1)}$

$= \frac{6 \pm \sqrt{-100}}{2} = \frac{6 \pm 10i}{2} = 3 \pm 5i$ (same 2 roots)

Example 7

Find a cubic equation that has roots $(1-\sqrt{6})$ and $-\frac{3}{2}$. $\longrightarrow (x + \frac{3}{2}) \to (2x+3)$

(conjugate thm: $1+\sqrt{6}$ also a root)

① sum $= (1-\sqrt{6}) + (1+\sqrt{6})$

$= \frac{2}{1} = \frac{-b}{a}$ $a = 1$, $b = -2$

② product (2 roots → even) $= \frac{\text{const.}}{a} = (1-\sqrt{6})(1+\sqrt{6})$

$= 1 - 6 = -5 = \text{const.}$

Quad. Linear = cubic

③ $(1x^2 - 2x + -5)(2x+3)$

$2x^3 + -4x^2 - 10x + 3x^2 - 6x - 15$

④ $\boxed{2x^3 - x^2 - 16x - 15 = f(x)}$

Section 2.7 Summary:

Use the conjugate roots to form a quadratic by finding the sum for 'a' and 'b' values (sum $= -\frac{b}{a}$) and find the product for the constant.

If cubic, multiply the quadratic piece by the 3rd root's factor; combine like terms to get $ax^3 + bx^2 + cx + d = f(x)$.